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A Characterization of Reflexive Spaces by Means of Continuous Approximate Selections for Metric Projections

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Reflexive spaces are characterized with the help of metric projections which possess a continuity property similar to n-lower semi-continuity and admit continuous ε-approximate selections. An example showing that almost lower semi-continuity of a metric projection is not sufficient for the existence of a continuous selection is constructed.

1 INTRODUCTION

Let (X, τ) be a topological space, and (Y, d) a metric space. A mapping $F: X \to 2^Y$ which associates with every $x \in X$ a non-empty subset $F(x)$ of Y is said to be lower semi-continuous (l.s.c.) (respectively, upper semi-continuous (u.s.c.)) if, for each open set $\mathscr{U}$ in Y, the set $\{x \in X: F(x) \cap \mathscr{U} \neq \varnothing\}$ (respectively, the set $\{x \in X: F(x) \subset \mathscr{U}\}$) is open in X. A mapping $f: X \to Y$ is a selection for F if, for each $x \in X$, $f(x) \in F(x)$.

One of the most celebrated results on the existence of continuous selections is the following theorem of Michael [11]: If X is a paracompact (e.g., metric) space and $F: X \to 2^Y$ is l.s.c. and has closed convex images, then F admits a continuous selection. The key step in the proof of this theorem is the construction of continuous ε-approximate selections. For an arbitrary non-empty set $A \subseteq Y$ and $\varepsilon > 0$, let $B_\varepsilon(A)$ denote the union of open balls with radii equal to ε and centers running over A. A mapping $f: X \to Y$ is called an ε-approximate selection for $F: X \to 2^Y$ if for each x in X $f(x) \in B_\varepsilon(F(x))$.

In [7] Deutsch and Kenderov introduced two continuity properties for multivalued mappings and identified topologically those mappings which admit continuous ε-approximate selections.

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DEFINITION (Deutsch and Kenderov). A multivalued mapping F: $X \to 2^Y$ is said to be almost lower semi-continuous (a.l.s.c.) (resp. n-lower semi-continuous (n-l.s.c.)) at $x_0 \in X$ if for each $\varepsilon > 0$ there is a neighbourhood $\mathscr{U}$ of x_0 such that $\bigcap \{B_\varepsilon(F(x)) \neq \varnothing$: $x \in \mathscr{U}\}$ (resp. $\bigcap_{i=1}^n B_\varepsilon(F(x_i)) \neq \varnothing$ for each choice of n points $x_1, x_2, ..., x_n$ in $\mathscr{U}$). F is a.l.s.c. (resp. n-l.s.c.) if F is a.l.s.c. (resp. n-l.s.c.) at each point x of X.

For our purposes we give a slightly different

DEFINITION. A multivalued mapping F: $X \to 2^Y$ is said to be finite lower semi-continuous (f.l.s.c.) at x_0 if for each $\varepsilon > 0$ there is a neighbourhood $\mathscr{U}$ of x_0 such that for each finite set of points A in $\mathscr{U}$ $\bigcap_{x \in A} B_\varepsilon(F(x)) \neq \varnothing$. F is f.l.s.c. if F is f.l.s.c. at each point x of X.

One of the main results in [7] is the following

THEOREM (Deutsch and Kenderov). *Let X be a paracompact space and let Y be a normed linear space. Suppose F: $X \to 2^Y$ has convex images. Then F is a.l.s.c. if, and only if, for each $\varepsilon > 0$ F admits a continuous ε-approximate selection.*

The above theorem, as well as other topological results in [7], Deutsch and Kenderov apply to metric projections. Recall that a map P_M: $X \to 2^M$, where $M \subseteq X$ and X is normed, is referred to as the metric projection generated by M provided that for each $x \in X$

$$P_M(x) = \{y \in M\colon \|y - x\| = d(x, M)\},$$

where

$$d(x, M) = \inf\{\|x - z\|\colon z \in M\}$$

is the distance function generated by M. A set M is called proximinal if $P_M(x) \neq \varnothing$ for all x in X. It is well known that the proximinal sets are closed.

Various problems concerning existence or non-existence of continuous selections for metric projections are studied in [1–3, 7, 10, 12–15, 18, 19] and others. Closely related to [7] is the work of Beer [1]. We note that the notion of approximate selection in [2, 4, 5, 16, 17] bears a different meaning.

This paper is motivated by the work of Deutsch and Kenderov [7]. It contains two results. The first one gives a characterization of reflexivity: A Banach space X is reflexive if, and only if, for every equivalent norm in X every f.l.s.c. metric projection generated by a proximinal subset of X has continuous ε-approximate selections for each $\varepsilon > 0$. The second result shows that almost lower semi-continuity of a metric projection does not imply existence of a continuous selection, even for finite dimensions: In a five-dimensional Minkowskian space there is an a.l.s.c. metric projection, generated by a three-dimensional subspace, which fails to possess a continuous selection.

2. The Main Result

THEOREM 1. *Let X be a non-reflexive Banach space and $M \subseteq X$ be a closed subspace with* $\operatorname{codim}(M) = 2$. *Then there is an equivalent renorming of X such that M is proximinal and the metric projection $P_M\colon X \to 2^M$ is finite lower semi-continuous but not almost lower semi-continuous.*

Proof. Since M is closed and $\operatorname{codim}(M) = 2$, then M is non-reflexive itself, and X is isomorphic to $\mathbb{R}^2 \times M$. We will define an equivalent norm in the space $Z := \mathbb{R}^2 \times M$. Suppose $f \in M^*$ is a bounded linear functional with $\|f\| = 1$ which does not achieve its supremum on the closed unit ball $U(M)$. The existence of such a functional is ensured by the theorem of James [9].

Consider the sets

$$C = \{(r, s, \eta) \in \mathbb{R} \times \mathbb{R} \times M\colon s = \langle f, \eta \rangle,\ r^2 + s^2 \leqslant 1,\ \|\eta\|_M \leqslant 1\}$$

$$D = \{(0, t, \eta) \in \mathbb{R} \times \mathbb{R} \times M\colon |t| \leqslant 1,\ \|\eta\|_M \leqslant 1\},$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between M and M^*. Obviously, C and D are closed convex bounded and symmetric. Designate by V the closed convex hull of $C \cup D$, i.e., $V = \overline{\text{co}}(C \cup D)$. Then V is a closed convex bounded and symmetric set. Also, it has non-empty interior: If C_1 is the set $\{(r, 0, 0)\colon |r| \leqslant 1\}$, then $2^{-1}(C_1 + D) = \{(r/2, t/2, \eta/2)\colon |r| \leqslant 1,\ |t| \leqslant 1,\ \|\eta\|_M \leqslant 1\}$ has non-empty interior. On the other hand the latter set is properly contained in V.

Now V viewed as a unit ball defines an equivalent norm $\|\cdot\|$ in Z. Let $P_M\colon Z \to 2^M$ be the metric projection generated by M with respect to the V-norm.

For arbitrary $q \in [0, 2\pi)$, let $a_q = (\cos q, \sin q, 0) \in Z$. Our next goal is to determine the set $P_M(a_q)$. Notice that the orthogonal projections of C and D over $\mathbb{R}^2$ are both contained in the circle $\{(r, s, 0) \in \mathbb{R}^2 \times M\colon r^2 + s^2 \leqslant 1\}$. Since it is closed, the orthogonal projection of V is in the same circle too. Therefore

$$d(a_q, M) \geqslant 1. \tag{1}$$

Denote by m_q the affine set $\{(\cos q, \sin q, \eta) \in Z\colon \eta \in M\}$, $q \in [0, 2\pi)$. It follows from (1) that m_q does not intersect the interior of V. If we show that $V \cap m_q \neq \emptyset$, then the formula $P_M(a_q) = (a_q + V) \cap M$ will take place. Towards this end, suppose first that $q \neq \pi/2$ and that $q \neq 3\pi/2$. In this situation $m_q \cap D = \emptyset$. Moreover, both sets are separated by the functional $(\cos q, \sin q, 0) \in Z^*$. So are m_q and C. In order to prove that $V \cap m_q = C \cap m_q$, we need the following

LEMMA 1. *Let C, D, and H be closed convex subsets of a normed space X, and let $g \in X^*$ be a bounded linear functional such that*

$$\sup\{\langle g, y\rangle : y \in D\} = \alpha < \beta = \inf\{\langle g, z\rangle : z \in H\}, \qquad \textit{and}$$
$$\sup\{\langle g, x\rangle : x \in C\} \leqslant \beta.$$

Then $H \cap \overline{\text{co}}(C \cup D) = H \cap C$.

Proof. Obviously $H \cap C \subseteq H \cap \overline{\text{co}}(C \cup D)$. Let $z \in H$, $z = \lim z_n$, $z_n = \lambda_n x_n + (1 - \lambda) y_n$ where $(x_n) \subseteq C$, $(y_n) \subseteq D$, $(\lambda_n) \subset [0, 1]$. Choose a convergent subsequence of (λ_n). With abuse of notation, let $\lambda_n \to \lambda_0$. Then we have

$$\begin{aligned}\beta &\leqslant \langle g, z\rangle \leqslant \lambda_0 \overline{\lim} \langle g, x_n\rangle + (1 - \lambda_0) \overline{\lim} \langle g, y_n\rangle \\ &\leqslant \lambda_0 \beta + (1 - \lambda_0)\alpha \leqslant \lambda_0 \beta + (1 - \lambda_0)\beta = \beta.\end{aligned}$$

This implies $\lambda_0 = 1$, whence $z = \lim x_n$. Therefore $z \in C$ because C is closed. The proof is completed.

By Lemma 1 $V \cap m_q = C \cap m_q$. So

$$\begin{aligned}V \cap m_q = \{(\cos q, \sin q, 0) \in Z &: \langle f, \eta\rangle \\ &= \sin q, \|\eta\|_M \leqslant 1\}, \quad q \neq \pi/2, 3\pi/2 \end{aligned} \tag{2}$$

The explicit form of $V \cap m_q$ convinces us that V and m_q have a nonempty intersection.

For $q = \pi/2$ we obtain

$$V \cap m_{\pi/2} \supseteq D \cap m_{\pi/2} = \{(0, 1, \eta) \in Z : \|\eta\|_M \leqslant 1\}. \tag{3}$$

Analogously, for $q = 3\pi/2$

$$V \cap m_{3\pi/2} \supseteq D \cap m_{3\pi/2} = \{(0, -1, \eta) \in Z : \|\eta\| \leqslant 1\}. \tag{4}$$

It is a routine matter to verify that $P_M(a_q) = (a_q + V) \cap M = a_q + V \cap (-m_q)$, whence by (2)–(4) we have

$$\begin{aligned}P_M(a_q) = \{(0, 0, \eta) \in \mathbb{R} \times \mathbb{R} \times M &: \langle f, \eta\rangle \\ &= -\sin q, \|\eta\|_M \leqslant 1\}, \qquad q \notin \{\pi/2, 3\pi/2\}\end{aligned} \tag{5}$$

as well as

$$P_M(a_{\pi/2}) \supseteq \{(0,0,\eta) \in Z\colon \|\eta\|_M \leqslant 1\} \tag{6}$$

and

$$P_M(a_{3\pi/2}) \supseteq \{(0,0,\eta) \in Z\colon \|\eta\|_M \leqslant 1\}. \tag{7}$$

In this way, for the points of the circumference $E = \{(\cos q, \sin q, 0) \in Z\colon q \in [0, 2\pi)\}$, $q \notin \{\pi/2, 3\pi/2\}$, the images of P_M correspond to the level-sets of f intersected by the closed unit ball $U(M)$, while for $q = \pi/2$ or $q = 3\pi/2$, $U(M)$ is contained in $P_M(a_q)$.

We claim now that the restriction of P_M over E is finite lower semi-continuous. The claim is almost obvious; however, for the sake of completeness, we give a demonstration in the particular case $q_0 = \pi/2$ (for arbitrary q the proof is similar).

Fix $\varepsilon > 0$ ($\varepsilon < \pi/2$) and take an open neighbourhood $\mathscr{U}$ in Z, $a_{q_0} \in \mathscr{U}$, such that for arbitrary $a_q = (\cos q, \sin q, 0)$, $a_q \in \mathscr{U}$, it follows that $|q - \pi/2| < \varepsilon$. Let $(a_{q_i})_{i=1}^n \in \mathscr{U}$ and k is an index satisfying $|q_k - \pi/2| = \min\{|q_i - \pi/2|\colon i = 1, 2, ..., n\}$. Suppose $q_k \neq \pi/2$ (the case $q_k = q_0$ is trivial) and take $y \in P_M(a_{q_k})$. Then $\langle f, y\rangle = -\sin q_k$. For each i choose λ_i, $0 < \lambda_i \leqslant 1$, such that $\langle f, \lambda_i y\rangle = -\sin q_i$, and define $y_i = \lambda_i y$. Since y_i belongs to $P_M(a_{q_i})$, we have the estimation

$$\begin{aligned} \|y - y_i\| &= (1-\lambda_i)\,\|y\| \leqslant 1 - \lambda_i \\ &= 1 - \frac{\sin q_i}{\sin q_k} < 1 - \sin q_i \leqslant |\pi/2 - q_i| < \varepsilon. \end{aligned}$$

Therefore $\bigcap_{i=1}^n B_\varepsilon(P_M(a_{q_i})) \neq \varnothing$, i.e., $P_{M|E}$ is f.l.s.c. at a_{q_0}. Our next lemma implies that P_M is everywhere f.l.s.c.

LEMMA 2. *Let $Z = (Y \times M, \|\cdot\|)$ be a product space of two Banach spaces Y and M, and let P_M be the metric projection generated by M (i.e., by $\{0\} \times M$). If for $E = \{(y, 0) \in Z\colon \|y\|_Y = 1\}$ the restriction map $P_{M|E}$ is a.l.s.c. (respectively f.l.s.c.) and has non-empty images, then so is P_M.*

Proof. For arbitrary $z \in Z$ the representation $z = \lambda y + m$ holds, where $\lambda \geqslant 0$, $y \in Y$, $\|y\|_Y = 1$, $m \in M$. We claim that

$$P_M(z) = m + \lambda \cdot P_M(y). \tag{8}$$

Designate the closed unit ball of Z by V and suppose $d(y, M) = r > 0$. Then $m + \lambda P_M(y) = m + \lambda(M \cap (y + rV)) = m + M \cap (\lambda y + \lambda r V) = M \cap (z + \lambda r V)$. Now since for any $k \in (0, r)$ $M \cap (y + kV) = \varnothing$, then $M \cap (z + \lambda k V) = \varnothing$. Therefore $d(z, M) = \lambda r$, which establishes the claim. In particular, (8) implies $P_M(z) \neq \varnothing$.

We next prove that P_M is a.l.s.c. at z_0 where z_0 is an arbitrary point in Z (the case of f.l.s.c. is treated analogously). If $z_0 = m_0 \in M$, then for each $z \in B_{\varepsilon/2}(m_0)$ and each $m \in P_M(z)$

$$\|m_0 - m\| \leqslant \|m_0 - z\| + \|z - m\| \leqslant 2 \cdot \|m_0 - z\| < \varepsilon,$$

whence $\bigcap\{B_\varepsilon(P_M(z))\colon \|z - z_0\| < \varepsilon/2\} \neq \emptyset$. So let $z_0 = \lambda y_0 + m_0$, $\lambda_0 > 0$, $\|y_0\|_Y = 1$, $m_0 \in M$. Since $P_{M|E}$ is a.l.s.c. at y_0, there exist $\delta > 0$ and $u_0 \in Z$ such that

$$u_0 \in \bigcap \{B_{\varepsilon/3\lambda_0}(P_M(y))\colon y \in E, \|y - y_0\| < \delta\}. \tag{9}$$

Obviously, we can always assume that $\|u_0\| > 0$. Consider the open neighbourhood of z_0

$$\mathscr{U} = \{\lambda y + m\colon |\lambda - \lambda_0| < \min\{\varepsilon/4\,\|u_0\|, \lambda_0/2\},$$
$$y \in E, \|y - y_0\| < \delta, m \in M, \|m - m_0\| < \varepsilon/4\}.$$

Suppose $z \in \mathscr{U}$, $z = \lambda y + m$, and take in (9) a point $u \in P_M(y)$ such that $\|u - u_0\| < \varepsilon/3\lambda_0$. According to (8) $\lambda u + m \in P_M(z)$. It follows from

$$\|\lambda u + m - \lambda_0 u_0 - m_0\| \leqslant \lambda\,\|u - u_0\| + |\lambda - \lambda_0| \cdot \|u_0\| + \|m - m_0\|$$
$$< \varepsilon\lambda/3\lambda_0 + \varepsilon/2 < \varepsilon$$

that $\lambda_0 u_0 + m_0 \in \bigcap\{B_\varepsilon(P_M(z))\colon z \in \mathscr{U}\}$, and this completes the proof.

In this way, for an arbitrary bounded linear functional $f \in M^*$ not achieving its norm, we defined an equivalent norm in Z with respect to which M is proximinal and P_M is f.l.s.c. Now f is chosen in a more sophisticated manner so that P_M fails to be a.l.s.c. In doing so we employ a theorem of James. But, before that, we make some explanatory remarks.

Suppose $(g_n) \subset M^*$ is a sequence of bounded linear functionals. Denote by $L(g_n)$ the set

$$\{w \in M^*\colon \underline{\lim}\langle g_n, x\rangle \leqslant \langle w, x\rangle \leqslant \overline{\lim}\langle g_n, x\rangle, \forall x \in M\}$$

and observe that $L(g_n)$ is non-empty. Indeed, the mapping $T\colon M \to l_\infty$, $T(x) = (\langle g_n, x\rangle)$, associates with each $x \in M$ a bounded sequence. If $\varphi \in l_\infty^*$ is a Banach limit, then $\underline{\lim}\langle g_n, x\rangle \leqslant \varphi(T(x)) \leqslant \overline{\lim}\langle g_n, x\rangle$ whence $w(\cdot) = \varphi(T(\cdot)) \in M^*$.

For arbitrary $f \in M^*$, $\|f\| = 1$, denote

$$S(f, \gamma) = \{x \in U(M)\colon \{f, x\rangle = \gamma\}, \quad 0 < \gamma < 1.$$

It follows from (5–7) and Lemma 2 that $P_{M|E}$ is a.l.s.c. at $(0, -1, 0) \in Z$ if, and only if,

$$\forall \varepsilon > 0\; \exists \gamma_0 \in (0, 1)\colon \bigcap_{\gamma \geqslant \gamma_0} B_\varepsilon(S(f, \gamma)) \neq \emptyset. \tag{10}$$

Using the next theorem we show that in a non-reflexive Banach space there exists a functional f which does not satisfy (10).

THEOREM (James [9], [8, p. 12]). *Suppose M is a non-reflexive Banach space. Then for $\theta \in (0, 1)$ and $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$, there exist α, $\theta \leqslant \alpha \leqslant 2$, and $(g_n) \subset M^*$, $\|g_n\| \leqslant 1$, such that each $w \in L(g_n)$ satisfies*

$$\left\| \sum_{n=1}^{\infty} \lambda_n (g_n - w) \right\| \leqslant \alpha \tag{11}$$

and

$$\left\| \sum_{n=1}^{k} \lambda_n (g_n - w) \right\| \leqslant \alpha \cdot \left(1 - \theta \cdot \sum_{n=k+1}^{\infty} \lambda_n \right), \quad k \in \mathbb{N}. \tag{12}$$

Pick $\theta \in (0, 1)$ and choose $\delta > 0$ so that $\delta < \theta^2/2$. If $\lambda_1 = 1 - \delta$, $\lambda_{n+1} = \delta \lambda_n$, then $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$, and according to the theorem of James there exist α, $\theta \leqslant \alpha \leqslant 2$, and $(g_n) \subset M^*$, $\|g_n\| \leqslant 1$, such that each w, $w \in L(g_n)$, satisfies (11) and (12).

For an arbitrary fixed functional w, $w \in L(g_n)$, take $f = \alpha^{-1} \sum_{n=1}^{\infty} \lambda_n (g_n - w)$ where $\|f\| = 1$. We claim that f does not satisfy (10). Assume the contrary. Then for $0 < \varepsilon < (\alpha\theta - 2\delta)/2 \cdot \delta/(1 - \delta)$ there are $x_\varepsilon \in U(M)$ and $\gamma_0 \in (0, 1)$ such that $x_\varepsilon \in B_\varepsilon(S(f, \gamma))$ whenever $\gamma \in (\gamma_0, 1)$. Since $\underline{\lim} \langle g_n, x_\varepsilon \rangle \leqslant \langle w, x_\varepsilon \rangle$, there is k so that

$$\langle g_k - w, x_\varepsilon \rangle < \alpha\theta - 2\delta.$$

Estimate $\langle f, x \rangle$ for $x \in U(M)$, $\|x - x_\varepsilon\| \leqslant \varepsilon$,

$$\begin{aligned}
\alpha \langle f, x \rangle &= \left\langle \sum_{n=1}^{k-1} \lambda_n (g_n - w), x \right\rangle + \lambda_k \langle g_k - w, x - x_\varepsilon \rangle \\
&\quad + \lambda_k \langle g_k - w, x_\varepsilon \rangle + \left\langle \sum_{n=k+1}^{\infty} \lambda_n (g_n - w), x \right\rangle \\
&\leqslant \left\| \sum_{n=1}^{k-1} \lambda_n (g_n - w) \right\| + 2\varepsilon \lambda_k \\
&\quad + (\alpha\theta - 2\delta)\, \lambda_k + 2 \cdot \sum_{n=k+1}^{\infty} \lambda_n \\
&\leqslant \alpha \cdot \left(1 - \theta \cdot \sum_{n=k}^{\infty} \lambda_n \right) + 2\varepsilon \lambda_k \\
&\quad + (\alpha\theta - 2\delta) \lambda_k + 2\delta \cdot \sum_{n=k}^{\infty} \lambda_n \\
&= \alpha - (\alpha\theta - 2\delta) \cdot \sum_{n=k+1}^{\infty} \lambda_n + 2\varepsilon \lambda_k .
\end{aligned}$$

Hence

$$\langle f, x\rangle \leqslant 1-c, \tag{13}$$

where

$$c=\alpha^{-1}\left[(\alpha\theta-2\delta)\cdot\sum_{n=k+1}^{\infty}\lambda_n-2\varepsilon\lambda_k\right].$$

Since ε is sufficiently small, then

$$c>\alpha^{-1}(\alpha\theta-2\delta)\cdot\left[\sum_{n=k+1}^{\infty}\lambda_n-\frac{\delta\lambda_k}{1-\delta}\right]=0.$$

It follows from (13) that $B_\varepsilon(x_\varepsilon)\cap S(f,\gamma)=\varnothing$ whenever $\gamma>1-c$ and this contradicts the choice of x_ε. Therefore f does not satisfy (10). Therefore P_M is not a.l.s.c. at $(0,-1,0)\in Z$. The proof of Theorem 1 is completed.

With the help of Theorem 1 and the theorem of Deutsch and Kenderov we give the following criterion for reflexivity:

THEOREM 2. *A Banach space X is reflexive if, and only if, for every equivalent renorming of X every finite lower semi-continuous metric projection generated by a convex proximinal subset of X admits a continuous ε-approximate selection for each $\varepsilon>0$.*

Proof. Necessity. Let $(X,\|\cdot\|)$ be reflexive. Suppose $|\cdot|$ is an equivalent norm and M is a convex subset of X which is proximinal with respect to $|\cdot|$. Suppose also that the metric projection $P_M\colon X\to 2^M$ is f.l.s.c. The Banach space $(X,|\cdot|)$ is reflexive and M is closed. We recall that in a reflexive space a convex set is proximinal if, and only if, it is closed. For arbitrary $x\in(X,|\cdot|)$ and $\varepsilon>0$ there exists a neighbourhood $\mathscr{U}$ of x such that $\bigcap_{i=1}^{n} B_{\varepsilon/2}(P_M(x_i))\neq\varnothing$ for each n and each choice of n points x_1, x_2, ..., x_n. Now the family $\{\overline{B_{\varepsilon/2}(P_M(z))}\colon z\in\mathscr{U}\}$, whose elements are weakly compact sets, has the finite intersection property and then it has a non-empty intersection. Therefore P_M is a.l.s.c. and according to the theorem of Deutsch and Kenderov P_M admits a continuous ε-approximate selection for each $\varepsilon>0$.

Sufficiency. Suppose X is non-reflexive. It follows from Theorem 1 that there exist an equivalent norm $|\cdot|$ and a convex proximinal set $M\subset X$ such that the metric projection $P_M\colon X\to 2^M$ is f.l.s.c. with respect to $|\cdot|$, but it lacks a.l.s.c. Apply the theoerem of Deutsch and Kenderov again, the sufficiency part, to prove that for some $\varepsilon>0$ P_M fails to admit a continuous ε-approximate selection.

3. Example of an a.l.s.c. Metric Projection, Generated by a Three-Dimensional Subspace of a Five-Dimensional Space, which Does Not Have a Continuous Selection

In the sequel S^n and B^n will stand for the unit sphere and the closed unit ball of the n-dimensional Euclidean space $\mathbb{R}^n$, respectively. The Euclidean norm is denoted by $|\cdot|$.

The following simple example of an a.l.s.c. u.s.c. mapping ϕ: $\mathbb{R} \to 2^{\mathbb{R} \times \mathbb{R}}$ which fails to admit a continuous selection has motivated our further considerations. The example is a modification of an analogous example due to Ch. Dangalchev [6]. Earlier examples of the same nature, but without upper semi-continuity involved, have been constructed by Pelant, c.f. [7], and Beer [1].

Suppose $(\theta_n)_{n=1}^{\infty}$ is a strictly decreasing sequence of positive reals so that $\lim_{n\to\infty} \theta_n = 0$. Define another sequence $(\omega_n)_{n=1}^{\infty}$ by $\omega_n = 2^{-1}(\theta_n + \theta_{n+1})$. The points P_n and Q_n have coordinates $(\omega_n, 1)$ and $(\theta_n, -1)$, respectively. Denote by Δ_n the triangle $[Q_n, P_n, Q_{n+1}]$ for $n = 1, 2, \ldots$. The mapping ϕ is defined as follows: For $x \geqslant 0$ $(x \leqslant \omega_1)$,

$$\phi(x) = \begin{cases} \Delta_n & \text{if } \ x = \omega_n \\ [Q_n, (x, 1)] & \text{if } \ \omega_n < x < \omega_{n-1} \\ [(0, -1), (0, 1)] & \text{if } \ x = 0, \end{cases}$$

and for $x < 0$ $\phi(x) = -\phi(-x)$.

It is easily checked that ϕ is a.l.s.c. Suppose f: $\mathbb{R} \to \mathbb{R}^2$ is a selection for ϕ which is continuous at both ω_n and $-\omega_n$. Then $f(\omega_n) = P_n$ and $f(-\omega_n) = -P_n$. On the other hand $\lim_{n\to\infty} P_n \neq (0, 0)$ and f cannot be continuous at 0.

We consider next a multivalued mapping $\tilde{\phi}$, in a certain sense similar to ϕ, which admits a mechanical interpretation: The images of $\tilde{\phi}$ might be viewed as sets of contact when a cylinder-like solid is rolling over a plane.

Denote $D_1 = \{(\cos\varphi,\ \sin\varphi,\ 1) \in \mathbb{R}^3\colon\ \varphi \in [0,\ 2\pi)\}$ and $D_2 = \{(\cos\varphi, \sin\varphi,\ -1) \in \mathbb{R}^3\colon \varphi \in [0, 2\pi)\}$. Let γ_1 and γ_2 be the planes carried by D_1 and D_2, respectively. Suppose (θ_n) and (ω_n) are two strictly decreasing sequences both defined for every integer $n \in \mathbb{Z}$ and satisfying $2\omega_n = \theta_n + \theta_{n+1}$. Moreover, suppose $\theta_0 = \pi/2$, $\theta_{-n} = \pi - \theta_n$, $\lim_{n\to-\infty} \theta_n = \pi$, $\lim_{n\to+\infty} \theta_n = 0$. Then $\omega_{-n} = \pi - \omega_n$, $\lim_{n\to-\infty} \omega_n = \pi$, $\lim_{n\to+\infty} \omega_n = 0$. Put $P_n = (\cos\omega_n, \sin\omega_n, 1)$ and $Q_n = (\cos\theta_n, \sin\theta_n, -1)$. We now define a three-dimensional convex body W by description of its surrounding surface Ξ (see Fig. 1).

The segments $[(1, 0, -1), (1, 0, 1)]$ and $[(-1, 0, -1), (-1, 0, 1)]$ are part of Ξ. So are the triangles $\Delta_n = [Q_n, P_n, Q_{n+1}]$. Let C_n be the cones, with vertices Q_n, generated by D_1. The conical sectors K_n which also

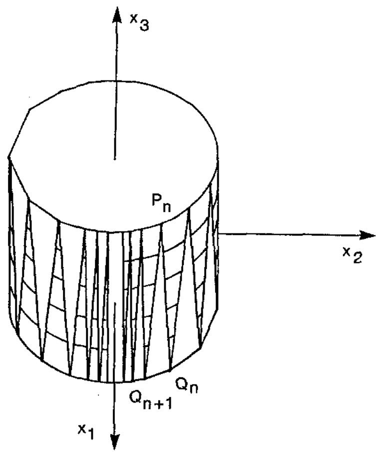

FIG. 1. The conical sectors are marked by dashed lines.

belong to Ξ are cut off from C_n by γ_1, γ_2 and the planes through Δ_{n-1} and Δ_n. Observe that for each n the plane supported by Δ_n meets γ_1 at a line l_n which is a tangent to D_1 at P_n. Hence the planes through two adjoint triangles Δ_{n-1} and Δ_n are tangent to the surface of C_n, the segments $[P_{n-1}, Q_n]$ and $[Q_n, P_n]$ being generatrices for C_n. To complete the definition of Ξ note that $X \in \Xi$ implies $-X \in \Xi$. Finally, define $W = \overline{\text{co}}\, \Xi$.

Thus we have a closed convex bounded and symmetric set with non-empty interior. Let $\|\cdot\|$ be the norm generated by W via the Minkowski functional. Denote by M the Minkowskian space $(\mathbb{R}^3, \|\cdot\|)$.

For each point $x \in S^3$, with coordinates (x_1, x_2, x_3), $|x_3| \neq 1$, define $\pi(x) = (x_1/\sqrt{x_1^2 + x_2^2},\ x_2/\sqrt{x_1^2 + x_2^2},\ 0)$, i.e., π projects x along the "meridian" on the "equator" $E = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon {}^2x_1 + {}^2x_2 = 1\}$.

It is clear from the definition of W that for every $y = (y_1, y_2, y_3) \in bdW$ with $|y_3| \neq 1$ there exists a uniquely determined normal vector $v(y) \in S^3$. Consider the set

$$r = \{v(y) \in S^3\colon y = (y_1, y_2, y_3),\ |y_3| \neq 1,\ \|y\| = 1\},$$

which is symmetric since W is symmetric itself. There is no difficulty in verifying that E is a homeomorphic image of r via π. Then r might be viewed as a parametric curve with a parameter φ, where φ is the oriented angle between the axis Ox_1 and $\pi(v(y))$.

Denote by $h_W(\cdot)$ the support function generated by W, i.e., $h_W(x) = \max\{\langle x, z\rangle\colon z \in W\}$. For each $x \in r$ let

$$F_x = \{y \in W\colon \langle x, y\rangle = h_W(x)\}.$$

Define $\tilde{\phi}$ as a composed map $E \to^{\pi^{-1}} r \to^{-F} W$, where $-F(x) = -(F_x)$ whenever $x \in r$. The images of $\tilde{\phi}$ are the "contact sets" of W and a plane "rolling" around W. Evidently, $\tilde{\phi}$ is a.l.s.c. The absence of a continuous selection for $\tilde{\phi}$ is shown in the same way as this was done for the mapping ϕ.

At the final stage of our construction we introduce a new norm in $\mathbb{R}^2 \times M$ such that the metric projection P_M: $\mathbb{R}^2 \times M \to 2^M$ restricted on the circumference $C = \{(\xi, 0) \in \mathbb{R}^2 \times M\colon |\xi| = 1\}$ is identical with $\tilde{\phi}$.

For arbitrary $x \in r$ let

$$G_x = \{(\xi, \eta) \in \mathbb{R}^2 \times M\colon \xi = \pi(x), \eta \in F_x\},$$

and define the new unit ball V by the formula

$$V = \overline{\text{co}}\left(\bigcup_{x \in r} G_x \cup \tfrac{1}{2} B^5\right).$$

Obviously, V is a closed bounded set with non-empty interior. It will suffice for symmetry to show that $\bigcup_{x \in r} G_x$ is symmetric. Indeed, if $(\xi, \eta) \in \bigcup_{x \in r} G_x$, there is $x \in r$ such that $\xi = \pi(x)$ and $\eta \in F_x$. Since W is symmetric, then $-\eta \in F_{-x}$. On the other hand $-\xi = \pi(-x)$ since r is symmetric. Hence $(-\xi, -\eta) \in G_{-x}$ and $-x \in r$. Thus V defines a norm in $\mathbb{R}^2 \times M$ which we also denote by $\|\cdot\|$.

Identifying in notation $\{0\} \times M$ with M, let P_M be the metric projection generated by the three-dimensional subspace M. We claim that

$$P_M(\xi, 0) = \{0\} \times F_{-x} \tag{14}$$

whenever $\xi \in \mathbb{R}^2$, $|\xi| = 1$ and $x = \pi^{-1}(\xi)$. The orthogonal projection along M maps V on B^2. For an arbitrary $\xi \in S^2$, denote $m_\xi = \{(\xi, \eta)\colon \eta \in M\}$. It is clear that $d((0, 0), m_\xi) \geq 1$. Suppose ξ is a fixed point on S^2 and $\xi = \pi(x)$. We prove next

$$G_x = m_\xi \cap V \tag{15}$$

The inclusion $G_x \subseteq m_\xi \cap V$ follows immediately. Conversely, if $(\xi, \eta) \in m_\xi \cap V$, then $|\xi| = 1$ and $(\xi, \eta) = \lim_{n \to \infty}(\xi_n, \eta_n)$ where $(\xi_n, \eta_n) \in \text{co}(\bigcup_{x \in r} G_x \cup \frac{1}{2} B^5)$. According to the theorem of Carathéodory $(\xi_n, \eta_n) = \sum_{i=1}^6 \lambda_{ni}(\xi_{ni}, \eta_{ni})$ where $0 \leq \lambda_{ni} \leq 1$, $\sum_{i=1}^6 \lambda_{ni} = 1$, $(\xi_{ni}, \eta_{ni}) \in \bigcup_{x \in r} G_x \cup \frac{1}{2} B^5$, $n = 1, 2, \ldots, i = 1, \ldots, 6$. We may assume, by passing to subsequences, that for every i $\lim_{n \to \infty} \lambda_{ni} = \lambda_{oi}$ and $\lim_{n \to \infty}(\xi_{ni}, \eta_{ni}) = (\xi_{oi}, \eta_{oi})$. So, with abuse of notation, we write

$$(\xi, \eta) = \sum_{i=1}^k \lambda_{oi}(\xi_{oi}, \eta_{oi}),\ \lambda_{oi} > 0, \quad \sum_{i=1}^k \lambda_{oi} = 1,\ k \leq 6.$$

Since $\xi = \sum_{i=1}^{k} \lambda_{oi} \xi_{oi}$ and $|\xi_{oi}| \leqslant 1$, we have from the strict convexity of B^2 that $\xi_{oi} = \xi$ for $i = 1, 2, ..., k$. Suppose i is a fixed index. If (ξ_{ni}, η_{ni}) were in $\frac{1}{2} B^5$ for infinitely many values for n, then ξ_{oi} would belong to $\frac{1}{2} B^5$ too. But this is incompatible with our choice of ξ. So for large n $(\xi_{ni}, \eta_{ni}) \in \bigcup_{x \in r} G_x$. Therefore there exist uniquely determined points $x_{ni} \in S^3$ such that $\xi_{ni} = \pi(x_{ni})$, $\eta_{ni} \in F_{x_{ni}}$ whence

$$\langle x_{ni}, \eta_{ni} \rangle = h_W(x_{ni}). \tag{16}$$

Since π is a homeomorphism, then $\lim_{n \to \infty} x_{ni} = \lim_{n \to \infty} \pi^{-1}(\xi_{ni}) = \pi^{-1}(\xi) \in r$. Taking $x = \pi^{-1}(\xi)$ and letting n go to infinity in (16), we obtain $\langle x, \eta_{oi} \rangle = h_W(x)$. Notice that $\eta_{oi} \in W$ since W is a closed set. On the other hand $x \in r$ and then $\eta_{oi} \in F_x$, $\xi = \pi(x)$. So $(\xi, \eta_{oi}) \in G_x$. It follows from the convexity of G_x that $(\xi, \eta) = (\xi, \sum_{i=1}^{k} \lambda_{oi} \eta_{oi}) \in G_x$. Thus (15) is established. In particular, (15) entails $d((0, 0), m_\xi) = 1$.

We proceed in determining the image of P_M at $(\xi, 0)$ for $\xi \in S^2$. As was shown above $d((\xi, 0), M) = 1$. Suppose $(z, y) \in [(\xi, 0) + V] \cap M$. Then $z = 0$ and $(z, y) = (\xi, 0) + (-\xi, y)$ whence $(-\xi, y) \in V$. So $(\xi, -y) \in m_\xi \cap V = G_x$ whenever $x = \pi^{-1}(\xi)$. We have $-y \in F_x$ which implies $y \in F_{-x}$. Thus $P_M(\xi, 0) \subseteq (0, F_{-x})$ for $\xi = \pi(x)$. Conversely, suppose $(0, y) \in (0, F_{-x})$ where $\xi = \pi(x)$. It follows from the representation $(0, y) = (\xi, 0) + (-\xi, y)$ that $(-\xi, y) \in G_{-x} \subseteq V$ because $-\xi \in \pi(-x)$. On the other hand obviously $(0, y) \in M$. The proof of our claim (14) is completed.

Finally, notice that since the restriction of P_M on the circumference C behaves like the mapping $\tilde{\phi}$, we need only apply Lemma 2 in order to make sure that P_M satisfies the required properties.

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